

## Radiative decays and the $SU(6)$ Lie algebra

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We present research on radiative decays of vector ( $J^{PC} = 1^{--}$ ) to pseudoscalar ( $J^{PC} = 0^{-+}$ ) particles ( $u, d, s, c, b, t$  quark system) using broken symmetry techniques in the infinite-momentum frame and equal-time commutation relations and the  $SU(6)$  Lie algebra, and conducted without ascribing any specific form to meson quark structure or intra-quark interactions. We utilize the physical electromagnetic current  $j_{em}^\mu(0)$  including its singlet  $U(1)$  term and focus on the  $SU(6)$  35-plet. We derive new relations involving the electromagnetic current (including its singlet — proportional to the  $SU(6)$  singlet). Remarkably, we find that the electromagnetic current singlet plays an intrinsic role in understanding the physics of radiative decays and that the charged and neutral  $\rho$  meson radiative decays into  $\pi\gamma$  are due entirely to the singlet term in  $j_{em}^\mu(0)$ . Although there is insufficient radiative decay experimental data available at this time, parametrization of possible predicted values of  $\Gamma(D^{*0} \rightarrow D^0\gamma)$  is made. For conciseness and self-containment, we compute all  $SU(6)$  Lie algebra simple roots, positive roots, weights and fundamental weights which allow the construction of all  $SU(6)$  representations. We also derive all nonzero  $SU(6)$  generator commutators and anticommutators — useful for further research on grand unified theories.

*Keywords:* Radiative decays of mesons; broken symmetry; infinite-momentum frame; equal-time commutation relations;  $SU(6)$  Lie algebra.

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### 1. Introduction

It is remarkable that in many cases observed particles appear to roughly fit into group-theoretical representation constructs which happen to be special unitary group representations. While these group-theoretical constructs obviously require that particles belonging to a particular representation all have the same mass, that is not what one observes in the real world — thus the need for *quark flavor broken symmetry group techniques*. To date, quantum chromodynamics (QCD)

based on Lagrangians (involving the addition of the Higgs field and other terms) invoking spontaneous symmetry breaking is the best theory for describing the real world, although lattice gauge models are making headway. As is well known, no theory capable of predicting and accommodating physical observations has yet been developed which incorporates the gravitational force. Indeed, although glueballs are predicted to exist in QCD, no *uncontrovertible* candidates have been found.

In this paper, we present research on radiative decays of vector ( $J^{PC} = 1^{--}$ ) to pseudoscalar ( $J^{PC} = 0^{-+}$ ) particles<sup>1</sup> which appear to belong — at least in part — (especially after application of broken symmetry techniques — infinite-momentum frame and asymptotic symmetry is discussed in Sec. 2) to specific flavor  $SU_F(6)$  representations. The representations of  $SU(N)$  — [special (determinant = unity), unitary] — classical Lie<sup>2</sup> groups are associated with the  $SU(N)$  classical, semisimple Lie algebras via linearly independent matrix operators  $V_a$  [the  $V_a$  are linear “charge” generators<sup>3</sup> — and  $V_a^\mu(x) = \bar{q}^i(x)(\lambda_a/2)_{ij}\gamma^\mu q^j(x)$  are the corresponding charge density operators ( $q$  represents the  $u, d, s, c, b, t$  quark system)] which act on the relevant vector space where (bilinear) commutators of the  $V_a$  are Lie products acting over the real number field. Each  $V_a$  is a Hermitian  $6 \times 6$  matrix for  $N = 6$  and there are  $6^2 - 1 = 35$   $V_a$ , where  $a = 1, \dots, 35$ .

In addition, we also introduce the singlet  $U(1)$  matrix  $V_0$  which is proportional to the identity matrix and commutes with all other generators and is *explicitly included in the physical electromagnetic current*  $j_{\text{em}}^\mu(0)$ . As we will discover in Sec. 3, *the singlet has an intrinsic role in understanding the physics of radiative decays*. Indeed, we introduce “generalized” Gell-Mann matrices (see Table 1)  $\lambda_a$  where  $V_a = \lambda_a/2$ . We will find that specific combinations of the  $V_a$  can be ultimately constructed which represent physical “raising” or “lowering” operators and we will label them using  $J^{PC} = 0^{-+}$  35-plet pseudoscalar particle names. Explicitly — *in the infinite-momentum frame* — (as we will demonstrate later in this paper) — for example, the *physical* vector charge  $V_{K^0}$  is  $V_{K^0} = V_6 + iV_7$  and the *physical* vector charge  $V_{\pi^\pm} = V_1 \pm iV_2$ . The  $\lambda_a$  satisfy the commutation algebra  $[(\lambda_a/2), (\lambda_b/2)] = i \sum_{c=1}^{N^2-1} f_{abc}(\lambda_c/2)$ , where the  $f_{abc}$  are structure constants (see Table 2) (we choose  $f_{abc}$  to be real and totally antisymmetric under permutations of the indices  $abc$  — we note that this can be done for  $SU(M)$  groups in general). For clarity and conciseness and self-containment, the  $SU(6)$  Lie algebra simple roots, positive roots, weights, fundamental weights, nonzero commutators (see Table 3), and nonzero anticommutators (see Table 5) are also determined which allow construction of all  $SU(6)$  representations. In Table 4, we also give all nonzero totally symmetric  $SU(6)$  tensors  $d_{ijk}$  useful in studying quark–gluon scattering and other processes. The  $d_{ijk}$  satisfy  $d_{ijk} = 2 \text{Tr}(V_i\{V_j, V_k\})$ .

Unless otherwise specified (or context specified), Lie algebras — (usually denoted by  $\mathfrak{su}(\mathfrak{n})$ ) — corresponding to Lie groups  $SU(N)$  — will just be denoted by  $SU(N)$ . Thus,  $SU(6)$  refers to the compact, analytic, continuous, semisimple Lie algebra for the Lie group  $SU(6)$ . Cartan<sup>4</sup> denotes  $SU(6)$  as  $A_5$  and  $SU(N)$



Table 2. Nonzero totally antisymmetric  $SU(6)$  structure constants  $f_{ijk}$ .

$i$	$j$	$k$	$f_{ijk}$	$i$	$j$	$k$	$f_{ijk}$	$i$	$j$	$k$	$f_{ijk}$	$i$	$j$	$k$	$f_{ijk}$
1	2	3	1	1	4	7	1/2	1	5	6	-1/2	1	9	12	1/2
1	10	11	-1/2	1	16	19	1/2	1	17	18	-1/2	1	25	28	1/2
1	26	27	-1/2	2	4	6	1/2	2	5	7	1/2	2	9	11	1/2
2	10	12	1/2	2	16	18	1/2	2	17	19	1/2	2	25	27	1/2
2	26	28	1/2	3	4	5	1/2	3	6	7	-1/2	3	9	10	1/2
3	11	12	-1/2	3	16	17	1/2	3	18	19	-1/2	3	25	26	1/2
3	27	28	-1/2	4	5	8	$\sqrt{3}/2$	4	9	14	1/2	4	10	13	-1/2
4	16	21	1/2	4	17	20	-1/2	4	25	30	1/2	4	26	29	-1/2
5	9	13	1/2	5	10	14	1/2	5	16	20	1/2	5	17	21	1/2
5	25	29	1/2	5	26	30	1/2	6	7	8	$\sqrt{3}/2$	6	11	14	1/2
6	12	13	-1/2	6	18	21	1/2	6	19	20	-1/2	6	27	30	1/2
6	28	29	-1/2	7	11	13	1/2	7	12	14	1/2	7	18	20	1/2
7	19	21	1/2	7	27	29	1/2	7	28	30	1/2	8	9	10	$1/(2\sqrt{3})$
8	11	12	$1/(2\sqrt{3})$	8	13	14	$-(1/\sqrt{3})$	8	16	17	$1/(2\sqrt{3})$	8	18	19	$1/(2\sqrt{3})$
8	20	21	$-(1/\sqrt{3})$	8	25	26	$1/(2\sqrt{3})$	8	27	28	$1/(2\sqrt{3})$	8	29	30	$-(1/\sqrt{3})$
9	10	15	$\sqrt{6}/3$	9	16	23	1/2	9	17	22	-1/2	9	25	32	1/2
9	26	31	-1/2	10	16	22	1/2	10	17	23	1/2	10	25	31	1/2
10	26	32	1/2	11	12	15	$\sqrt{6}/3$	11	18	23	1/2	11	19	22	-1/2
11	27	32	1/2	11	28	31	-1/2	12	18	22	1/2	12	19	23	1/2
12	27	31	1/2	12	28	32	1/2	13	14	15	$\sqrt{6}/3$	13	20	23	1/2
13	21	22	-1/2	13	29	32	1/2	13	30	31	-1/2	14	20	22	1/2
14	21	23	1/2	14	29	31	1/2	14	30	32	1/2	15	16	17	$1/(2\sqrt{6})$
15	18	19	$1/(2\sqrt{6})$	15	20	21	$1/(2\sqrt{6})$	15	22	23	$-3/(2\sqrt{6})$	15	25	26	$1/(2\sqrt{6})$
15	27	28	$1/(2\sqrt{6})$	15	29	30	$1/(2\sqrt{6})$	15	31	32	$-3/(2\sqrt{6})$	16	17	24	$\sqrt{10}/4$
16	25	34	1/2	16	26	33	-1/2	17	25	33	1/2	17	26	34	1/2
18	19	24	$\sqrt{10}/4$	18	27	34	1/2	18	28	33	-1/2	19	27	33	1/2
19	28	34	1/2	20	21	24	$\sqrt{10}/4$	20	29	34	1/2	20	30	33	-1/2
21	29	33	1/2	21	30	34	1/2	22	23	24	$\sqrt{10}/4$	22	31	34	1/2
22	32	33	-1/2	23	31	33	1/2	23	32	34	1/2	24	25	26	$1/(2\sqrt{10})$
24	27	28	$1/(2\sqrt{10})$	24	29	30	$1/(2\sqrt{10})$	24	31	32	$1/(2\sqrt{10})$	24	33	34	$-\sqrt{10}/5$
25	26	35	$\sqrt{15}/5$	27	28	35	$\sqrt{15}/5$	29	30	35	$\sqrt{15}/5$	31	32	35	$\sqrt{15}/5$
33	34	35	$\sqrt{15}/5$												

as  $A_{N-1}$ . It is known that the *lowest-dimensional*  $SU(N)$  representation is by  $N \times N$ , traceless matrix generators which we utilize in this paper. Data in this paper are taken from the Particle Data Group.<sup>5,a</sup>

<sup>a</sup>Particle charge conjugate states are utilized in this paper.





























Table 3 (Continued)

$216. [V_{35}, V_{T^0}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{3}{5}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $= \left(-\sqrt{\frac{3}{5}}V_{T^0}\right)$	$220. [V_{35}, V_{T_b^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{5}} & 0 \end{pmatrix}$ $= \left(-\sqrt{\frac{3}{5}}V_{T_b^+}\right)$
$217. [V_{35}, V_{T^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{3}{5}} & 0 & 0 & 0 & 0 \end{pmatrix}$ $= \left(-\sqrt{\frac{3}{5}}V_{T^+}\right)$	$221. [V_{T_b^-}, V_{T^0}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{B^-})$
$218. [V_{35}, V_{T_s^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{5}} & 0 & 0 & 0 \end{pmatrix}$ $= \left(-\sqrt{\frac{3}{5}}V_{T_s^+}\right)$	$222. [V_{T_b^-}, V_{T^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{B^0})$
$219. [V_{35}, V_{T_c^0}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{5}} & 0 & 0 & 0 \end{pmatrix}$ $= \left(-\sqrt{\frac{3}{5}}V_{T_c^0}\right)$	$223. [V_{T_b^-}, V_{T_s^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{B_s^0})$
$224. [V_{T_b^-}, V_{T_c^0}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{B_c^-})$	$225. [V_{T_b^-}, V_{T_b^+}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ $= \left(-2\sqrt{\frac{2}{5}}V_{24} + 2\sqrt{\frac{3}{5}}V_{35}\right)$

### 1.1. More about the $SU(6)$ Lie algebra

The defining bilinear operation — the commutator  $[,]$  — involving the structure constants and which determines the Lie algebra and a generator scalar product is given by

$$[V_a, V_b] = i \sum_{c=1}^{N^2-1} f_{abc} V_c, \quad (1a)$$

where  $a, b = 1, 2, \dots, N^2 - 1$

$$\text{Tr}[V_a V_b] = \frac{1}{2} \delta_{ab}, \quad (1b)$$

$$f_{abc} = -i2 \text{Tr}([V_a, V_b] V_c). \quad (1c)$$

Table 4. Nonzero totally symmetric  $SU(6)$  structure constant related tensors  $d_{ijk}$ .

$i$	$j$	$k$	$d_{ijk}$	$i$	$j$	$k$	$d_{ijk}$	$i$	$j$	$k$	$d_{ijk}$	$i$	$j$	$k$	$d_{ijk}$
0	$j$	$k$	$\delta_{jk}/\sqrt{3}$	1	1	8	$1/\sqrt{3}$	1	1	15	$1/\sqrt{6}$	1	1	24	$1/\sqrt{10}$
1	1	35	$1/\sqrt{15}$	1	4	6	$1/2$	1	5	7	$1/2$	1	9	11	$1/2$
1	10	12	$1/2$	1	16	18	$1/2$	1	17	19	$1/2$	1	25	27	$1/2$
1	26	28	$1/2$	2	2	8	$1/\sqrt{3}$	2	2	15	$1/\sqrt{6}$	2	2	24	$1/\sqrt{10}$
2	2	35	$1/\sqrt{15}$	2	4	7	$-1/2$	2	5	6	$1/2$	2	9	12	$-1/2$
2	10	11	$1/2$	2	16	19	$-1/2$	2	17	18	$1/2$	2	25	28	$-1/2$
2	26	27	$1/2$	3	3	8	$1/\sqrt{3}$	3	3	15	$1/\sqrt{6}$	3	3	24	$1/\sqrt{10}$
3	3	35	$1/\sqrt{15}$	3	4	4	$1/2$	3	5	5	$1/2$	3	6	6	$-1/2$
3	7	7	$-1/2$	3	9	9	$1/2$	3	10	10	$1/2$	3	11	11	$-1/2$
3	12	12	$-1/2$	3	16	16	$1/2$	3	17	17	$1/2$	3	18	18	$-1/2$
3	19	19	$-1/2$	3	25	25	$1/2$	3	26	26	$1/2$	3	27	27	$-1/2$
3	28	28	$-1/2$	4	4	8	$-1/(2\sqrt{3})$	4	4	15	$1/\sqrt{6}$	4	4	24	$1/\sqrt{10}$
4	4	35	$1/\sqrt{15}$	4	9	13	$1/2$	4	10	14	$1/2$	4	16	20	$1/2$
4	17	21	$1/2$	4	25	29	$1/2$	4	26	30	$1/2$	5	5	8	$-1/(2\sqrt{3})$
5	5	15	$1/\sqrt{6}$	5	5	24	$1/\sqrt{10}$	5	5	35	$1/\sqrt{15}$	5	9	14	$-1/2$
5	10	13	$1/2$	5	16	21	$-1/2$	5	17	20	$1/2$	5	25	30	$-1/2$
5	26	29	$1/2$	6	6	8	$-1/(2\sqrt{3})$	6	6	15	$1/\sqrt{6}$	6	6	24	$1/\sqrt{10}$
6	6	35	$1/\sqrt{15}$	6	11	13	$1/2$	6	12	14	$1/2$	6	18	20	$1/2$
6	19	21	$1/2$	6	27	29	$1/2$	6	28	30	$1/2$	7	7	8	$-1/(2\sqrt{3})$
7	7	15	$1/\sqrt{6}$	7	7	24	$1/\sqrt{10}$	7	7	35	$1/\sqrt{15}$	7	11	14	$-1/2$
7	12	13	$1/2$	7	18	21	$-1/2$	7	19	20	$1/2$	7	27	30	$-1/2$
7	28	29	$1/2$	8	8	8	$-(1/\sqrt{3})$	8	8	15	$1/\sqrt{6}$	8	8	24	$1/\sqrt{10}$
8	8	35	$1/\sqrt{15}$	8	9	9	$1/(2\sqrt{3})$	8	10	10	$1/(2\sqrt{3})$	8	11	11	$1/(2\sqrt{3})$
8	12	12	$1/(2\sqrt{3})$	8	13	13	$-(1/\sqrt{3})$	8	14	14	$-(1/\sqrt{3})$	8	16	16	$1/(2\sqrt{3})$
8	17	17	$1/(2\sqrt{3})$	8	18	18	$1/(2\sqrt{3})$	8	19	19	$1/(2\sqrt{3})$	8	20	20	$-(1/\sqrt{3})$
8	21	21	$-(1/\sqrt{3})$	8	25	25	$1/(2\sqrt{3})$	8	26	26	$1/(2\sqrt{3})$	8	27	27	$1/(2\sqrt{3})$
8	28	28	$1/(2\sqrt{3})$	8	29	29	$-(1/\sqrt{3})$	8	30	30	$-(1/\sqrt{3})$	9	9	15	$-(1/\sqrt{6})$
9	9	24	$1/\sqrt{10}$	9	9	35	$1/\sqrt{15}$	9	16	22	$1/2$	9	17	23	$1/2$
9	26	32	$1/2$	10	10	15	$-(1/\sqrt{6})$	10	10	24	$1/\sqrt{10}$	10	10	35	$1/\sqrt{15}$
10	16	23	$-1/2$	10	17	22	$1/2$	10	25	32	$-1/2$	10	26	31	$1/2$
11	11	15	$-(1/\sqrt{6})$	11	11	24	$1/\sqrt{10}$	11	11	35	$1/\sqrt{15}$	11	18	22	$1/2$
11	19	23	$1/2$	11	27	31	$1/2$	11	28	32	$1/2$	12	12	15	$-(1/\sqrt{6})$
12	12	24	$1/\sqrt{10}$	12	12	35	$1/\sqrt{15}$	12	18	23	$-1/2$	12	19	22	$1/2$
12	27	32	$-1/2$	12	28	31	$1/2$	13	13	15	$-(1/\sqrt{6})$	13	13	24	$1/\sqrt{10}$
13	13	35	$1/\sqrt{15}$	13	20	22	$1/2$	13	21	23	$1/2$	13	29	31	$1/2$
13	30	32	$1/2$	14	14	15	$-(1/\sqrt{6})$	14	14	24	$1/\sqrt{10}$	14	14	35	$1/\sqrt{15}$
14	20	23	$-1/2$	14	21	22	$1/2$	14	29	32	$-1/2$	14	30	31	$1/2$
15	15	15	$-2/\sqrt{6}$	15	15	24	$1/\sqrt{10}$	15	15	35	$1/\sqrt{15}$	15	16	16	$1/(2\sqrt{6})$
15	17	17	$1/(2\sqrt{6})$	15	18	18	$1/(2\sqrt{6})$	15	19	19	$1/(2\sqrt{6})$	15	20	20	$1/(2\sqrt{6})$
15	21	21	$1/(2\sqrt{6})$	15	22	22	$-3/(2\sqrt{6})$	15	23	23	$-3/(2\sqrt{6})$	15	25	25	$1/(2\sqrt{6})$
15	26	26	$1/(2\sqrt{6})$	15	27	27	$1/(2\sqrt{6})$	15	28	28	$1/(2\sqrt{6})$	15	29	29	$1/(2\sqrt{6})$
15	30	30	$1/(2\sqrt{6})$	15	31	31	$-3/(2\sqrt{6})$	15	32	32	$-3/(2\sqrt{6})$	16	16	24	$-3/(2\sqrt{10})$
16	16	35	$1/\sqrt{15}$	16	25	33	$1/2$	16	26	34	$1/2$	17	17	24	$-3/(2\sqrt{10})$
17	17	35	$1/\sqrt{15}$	17	25	34	$-1/2$	17	26	33	$1/2$	18	18	24	$-3/(2\sqrt{10})$
18	18	35	$1/\sqrt{15}$	18	27	33	$1/2$	18	28	34	$1/2$	19	19	24	$-3/(2\sqrt{10})$
19	19	35	$1/\sqrt{15}$	19	27	34	$-1/2$	19	28	33	$1/2$	20	20	24	$-3/(2\sqrt{10})$
20	20	35	$1/\sqrt{15}$	20	29	33	$1/2$	20	30	34	$1/2$	21	21	24	$-3/(2\sqrt{10})$
21	21	35	$1/\sqrt{15}$	21	29	34	$-1/2$	21	30	33	$1/2$	22	22	24	$-3/(2\sqrt{10})$
22	22	35	$1/\sqrt{15}$	22	31	33	$1/2$	22	32	34	$1/2$	23	23	24	$-3/(2\sqrt{10})$
23	23	35	$1/\sqrt{15}$	23	31	34	$-1/2$	23	32	33	$1/2$	24	24	24	$-3/\sqrt{10}$
24	24	35	$1/\sqrt{15}$	24	25	25	$1/(2\sqrt{10})$	24	26	26	$1/(2\sqrt{10})$	24	27	27	$1/(2\sqrt{10})$
24	28	28	$1/(2\sqrt{10})$	24	29	29	$1/(2\sqrt{10})$	24	30	30	$1/(2\sqrt{10})$	24	31	31	$1/(2\sqrt{10})$
24	32	32	$1/(2\sqrt{10})$	24	33	33	$-\sqrt{2/5}$	24	34	34	$-\sqrt{2/5}$	25	25	35	$-2/\sqrt{15}$
26	26	35	$-2/\sqrt{15}$	27	27	35	$-2/\sqrt{15}$	28	28	35	$-2/\sqrt{15}$	29	29	35	$-2/\sqrt{15}$
30	30	35	$-2/\sqrt{15}$	31	31	35	$-2/\sqrt{15}$	32	32	35	$-2/\sqrt{15}$	33	33	35	$-2/\sqrt{15}$
34	34	35	$-2/\sqrt{15}$	35	35	35	$-4/\sqrt{15}$								







































Table 5 (Continued)

299. $\{V_{T_b^-}, V_{T^+}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{\bar{B}^0})$	304. $\{V_{T^0}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left(\frac{V_{T^0}}{\sqrt{3}}\right)$
300. $\{V_{T_b^-}, V_{T_s^+}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{\bar{B}_s^0})$	305. $\{V_{T^+}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix} = \left(\frac{V_{T^+}}{\sqrt{3}}\right)$
301. $\{V_{T_b^-}, V_{T_c^0}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (V_{B_c^-})$	306. $\{V_{T_s^+}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{pmatrix} = \left(\frac{V_{T_s^+}}{\sqrt{3}}\right)$
302. $\{V_{T_b^-}, V_{T_b^+}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \left(\frac{2V_0}{\sqrt{3}} - 2\sqrt{\frac{2}{5}}V_{24} - \frac{4V_{35}}{\sqrt{15}}\right)$	307. $\{V_{T_c^0}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \end{pmatrix} = \left(\frac{V_{T_c^0}}{\sqrt{3}}\right)$
303. $\{V_{T_b^-}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left(\frac{V_{T_b^-}}{\sqrt{3}}\right)$	308. $\{V_{T_b^+}, V_0\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} = \left(\frac{V_{T_b^+}}{\sqrt{3}}\right)$

The structure constants and Lie generators have been constructed so that *they remain the same for*  $SU(n - m)$ , where  $n > m$  ( $n, m$  are positive integers). In general, commutators of Lie group generators (see Table 3) are themselves linear combinations of these same generators and the generator algebra is called a Lie algebra.<sup>6-14</sup> If the group has  $r$  (group order)  $\equiv N^2 - 1$  generators, then there are  $(1/2)(r - 1)r = (1/2)(N^2 - 2)(N^2 - 1)$  possible generator commutation relations. The rank  $l$  of a Lie group is equal to the *maximum* number of generators (linear) which *mutually commute*. There also exist  $l$  *non-linear* Casimir operators  $C_i = \sum_{j=1}^r a_{ji} V_j^{i+1}$ ,  $i = 1, 2, \dots, l$ , which commute with *all* of the algebra generators. Following primarily the notation of Lichtenberg<sup>9</sup> but also others:<sup>10-20</sup> — the *mutually commuting* generators are conventionally denoted by  $H_i$  ( $i = 1, 2, \dots, l$ ), where  $H_i = H_i^\dagger$ , and  $[H_i, H_j] = 0$  ( $i, j = 1, 2, \dots, l$ ). So  $H_1 = V_3 = \frac{\lambda_3}{2}$ ,  $H_2 = V_8 = \frac{\lambda_8}{2}$ ,  $H_3 = V_{15} = \frac{\lambda_{15}}{2}$ ,  $H_4 = V_{24} = \frac{\lambda_{24}}{2}$ ,  $H_5 = V_{35} = \frac{\lambda_{35}}{2}$  and  $\mathbf{H} \equiv (H_1, H_2, H_3, H_4, H_5)$ . The  $l$  (maximal number of *mutually commuting*

*Hermitian generators*) and  $H_i$  (*Cartan generators*) are the basis for what is called the *Cartan subalgebra* and constitute a linear space. The  $H_i$  are Cartan ( $A_5$ ) generators and the rank of the traceless, semisimple, compact Lie algebra of the *classical* group  $SU(6)$  is  $(6 - 1) = 5 =$  number of  $H_i$ 's.

## 1.2. Roots and weights

Given the generators  $V_a$ , one can construct  $A = \sum_{j=1}^r a_j V_j$  and the eigenvalue equation  $[A, X] = \rho X$ , where  $X$  is some linear combination of the  $V_j$ , then one can derive the secular equation (polynomial of degree  $r$ ) for the  $(r - l)$  eigenvalues called *roots*  $\rho$ , namely  $\det(\sum_{i=1}^r a_i f_{ijk} - \rho \delta_{jk}) = 0$ . For semisimple Lie groups (which includes  $SU(N)$ ), Cartan has shown that there are  $r$  independent eigenvectors (even if there exist degenerate roots for  $\rho = 0$ ) and the multiplicity of these degenerate roots is equal to the rank  $l$ .

We define (Cartan–Weyl formalism)  $l = 5$  generators (the  $H_i$  mentioned above),  $(r - l) = 30$  remaining generators,  $E_\alpha \equiv V_\alpha$ , where  $\alpha = (\pi^\pm, K^\pm, K^0, \bar{K}^0, \bar{D}^0, D^0, D^\pm, D_s^\pm, B^0, \bar{B}^0, B^\pm, B_s^0, \bar{B}_s^0, B_c^\pm, T^\pm, T^0, \bar{T}^0, T_s^\pm, T_c^0, \bar{T}_c^0, T_b^\pm)$ , six-dimensional basis states (vectors  $u_i$ ), the diagonal vector operator  $\mathbf{H}$ , and six five-dimensional *weights* (*convention dependent*)  $\mathbf{m}(i)$  of  $SU(6)$ :

$$H_i = H_i^\dagger, \quad [H_i, H_j] = 0, \quad (2a)$$

$$\text{Tr}[H_i, H_j] = \frac{1}{2} \delta_{ij}, \quad \mathbf{H} \equiv (H_1, H_2, H_3, H_4, H_5),$$

$$u_i = \begin{pmatrix} \delta_i^1 \\ \delta_i^2 \\ \delta_i^3 \\ \delta_i^4 \\ \delta_i^5 \\ \delta_i^6 \end{pmatrix} \Rightarrow u_1 = |u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2b)$$

$$u_2 = |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, u_6 = |t\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{H}u_i = \mathbf{m}(i)u_i, \quad \mathbf{m}(i) = (m_1, m_2, m_3, m_4, m_5, m_6), \quad (2c)$$

$$\mathbf{m}(1) = (H_1 u, H_2 u, H_3 u, H_4 u, H_5 u), \quad (2d)$$

$$\mathbf{m}(2) = (H_1 d, H_2 d, H_3 d, H_4 d, H_5 d),$$

$$\mathbf{m}(3) = (H_1s, H_2s, H_3s, H_4s, H_5s), \quad (2e)$$

$$\mathbf{m}(4) = (H_1c, H_2c, H_3c, H_4c, H_5c),$$

$$\mathbf{m}(5) = (H_1b, H_2b, H_3b, H_4b, H_5b), \quad (2f)$$

$$\mathbf{m}(6) = (H_1t, H_2t, H_3t, H_4t, H_5t).$$

Using Eqs. (2), Tables 1 and 3, we obtain

$$\mathbf{m}(1) = \left( \frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{15}} \right), \quad (3a)$$

$$\mathbf{m}(2) = \left( -\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{15}} \right),$$

$$\mathbf{m}(3) = \left( 0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{15}} \right), \quad (3b)$$

$$\mathbf{m}(4) = \left( 0, 0, -\frac{3}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{15}} \right),$$

$$\mathbf{m}(5) = \left( 0, 0, 0, -\frac{2}{\sqrt{10}}, \frac{1}{2\sqrt{15}} \right), \quad (3c)$$

$$\mathbf{m}(6) = \left( 0, 0, 0, 0, -\frac{5}{2\sqrt{15}} \right),$$

$$\mathbf{m}(i) \cdot \mathbf{m}(j) = -\frac{1}{2 * 6} + \frac{1}{2} \delta_{ij}, \quad (3d)$$

$$\sum_{i=1}^{i=6} \mathbf{m}(i) = 0. \quad (3e)$$

So the  $j$ th component of  $\mathbf{m}(k)$  is given by

$$m_j(k) = \begin{cases} [2j(j+1)]^{-1/2} & k < j+1, \\ -j[2j(j+1)]^{-1/2} & k = j+1, \\ 0 & k > j+1. \end{cases} \quad (4)$$

The eigenvalues  $\mathbf{m}(i)$  form a 5-simplex hexateron<sup>11,21</sup> and are the *weights of the first fundamental representation* of  $SU(6)$  spanning a  $l$ -dimensional vector *weight-space*. We use the convention that  $\mathbf{m}$  is higher than  $\mathbf{m}'$  if the last component of the vector  $\mathbf{m} - \mathbf{m}'$  is positive — if that is zero, one moves to the next component and so on. Thus, the (eigenvalues)  $\mathbf{m}(i)$  in Eq. (3) are already ordered. There exist four other fundamental representations of  $SU(6)$ , however one can construct the entire Lie algebra from the first fundamental representation. The positive roots  $\alpha_i$  are

$$\mathbf{m}(i) - \mathbf{m}(j) \quad \text{for } i < j \quad (i, j) = 1, \dots, 6. \quad (5)$$

There are 15 positive roots. We now introduce some helpful new notation:

$$\left. \begin{aligned}
 & \left( \begin{array}{cccccc}
 V_3 & V_1^2 & V_1^3 & V_1^4 & V_1^5 & V_1^6 \\
 V_2^1 & V_8 & V_2^3 & V_2^4 & V_2^5 & V_2^6 \\
 V_3^1 & V_3^2 & V_{15} & V_3^4 & V_3^5 & V_3^6 \\
 V_4^1 & V_4^2 & V_4^3 & V_{24} & V_4^5 & V_4^6 \\
 V_5^1 & V_5^2 & V_5^3 & V_5^4 & V_{35} & V_5^6 \\
 V_6^1 & V_6^2 & V_6^3 & V_6^4 & V_6^5 & V_0
 \end{array} \right) \\
 & \equiv \left( \begin{array}{cccccc}
 V_3 & V_{\pi^+} & V_{K^+} & V_{\bar{D}^0} & V_{B^+} & V_{T^0} \\
 V_{\pi^-} & V_8 & V_{K^0} & V_{D^-} & V_{B^0} & V_{T^-} \\
 V_{K^-} & V_{\bar{K}^0} & V_{15} & V_{D_s^-} & V_{B_s^0} & V_{T_s^-} \\
 V_{D^0} & V_{D^+} & V_{D_s^+} & V_{24} & V_{B_c^+} & V_{T_c^0} \\
 V_{B^-} & V_{\bar{B}^0} & V_{B_s^0} & V_{B_c^-} & V_{35} & V_{T_b^-} \\
 V_{T^0} & V_{T^+} & V_{T_s^+} & V_{T_c^0} & V_{T_b^+} & V_0
 \end{array} \right) \\
 & = \left( \begin{array}{cccccc}
 V_3 & V_1 + iV_2 & V_4 + iV_5 & V_9 + iV_{10} & V_{16} + iV_{17} & V_{25} + iV_{26} \\
 V_1 - iV_2 & V_8 & V_6 + iV_7 & V_{11} + iV_{12} & V_{18} + iV_{19} & V_{27} + iV_{28} \\
 V_4 - iV_5 & V_6 - iV_7 & V_{15} & V_{13} + iV_{14} & V_{20} + iV_{21} & V_{29} + iV_{30} \\
 V_9 - iV_{10} & V_{11} - iV_{12} & V_{13} - iV_{14} & V_{24} & V_{22} + iV_{23} & V_{31} + iV_{32} \\
 V_{16} - iV_{17} & V_{18} - iV_{19} & V_{20} - iV_{21} & V_{22} - iV_{23} & V_{35} & V_{33} + iV_{34} \\
 V_{25} - iV_{26} & V_{27} - iV_{28} & V_{29} - iV_{30} & V_{31} - iV_{32} & V_{33} - iV_{34} & V_0
 \end{array} \right) \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \left( \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right) \begin{array}{l} \\ \\ \\ \\ \\ \end{array} V_i^j \\
 & [i = \text{row index}, j = \text{column index}], \\
 & V_1^1 \equiv V_3 + \frac{1}{\sqrt{3}}V_8 + \frac{1}{\sqrt{6}}V_{15} + \frac{1}{\sqrt{10}}V_{24} + \frac{1}{\sqrt{15}}V_{35}, \\
 & V_2^2 \equiv -V_3 + \frac{1}{\sqrt{3}}V_8 + \frac{1}{\sqrt{6}}V_{15} + \frac{1}{\sqrt{10}}V_{24} + \frac{1}{\sqrt{15}}V_{35}, \\
 & V_3^3 \equiv -\frac{2}{\sqrt{3}}V_8 + \frac{1}{\sqrt{6}}V_{15} + \frac{1}{\sqrt{10}}V_{24} + \frac{1}{\sqrt{15}}V_{35}, \\
 & V_4^4 \equiv \frac{3}{\sqrt{6}}V_{15} + \frac{1}{\sqrt{10}}V_{24} + \frac{1}{\sqrt{15}}V_{35}, \\
 & V_5^5 \equiv -\frac{4}{\sqrt{10}}V_{24} + \frac{1}{\sqrt{15}}V_{35}, \quad V_6^6 \equiv \frac{5}{\sqrt{15}}V_{35}, \\
 & \text{and} \\
 & V_3 = \frac{1}{2}(V_1^1 - V_2^2), \quad V_8 = \frac{1}{2\sqrt{3}}(V_1^1 + V_2^2 - 2V_3^3), \\
 & V_{15} = \frac{1}{2\sqrt{6}}(V_1^1 + V_2^2 + V_3^3 - 3V_4^4), \\
 & V_{24} = \frac{1}{2\sqrt{10}}(V_1^1 + V_2^2 + V_3^3 + V_4^4 - 4V_5^5), \\
 & V_{35} = \frac{1}{2\sqrt{15}}(V_1^1 + V_2^2 + V_3^3 + V_4^4 + V_5^5 - 5V_6^6).
 \end{aligned} \right\} \quad (6)$$

We also have

$$[\mathbf{H}, E_\alpha] = [\mathbf{H}, V_\alpha] = \rho(\alpha)V_\alpha, \quad (7)$$

where the  $\rho(\alpha)$  are  $l$ -dimensional root vectors spanning a  $(r - l) = 30$ -dimensional root-space.

We can extract all roots  $\rho(\alpha)$  from the nonzero commutation relations given in Table 3 and using Eq. (7). We define a root as positive if its last component is positive — otherwise it is negative. In addition, it can be shown that  $\rho(\alpha^\dagger) = -\rho(\alpha)$  so that for instance  $\rho(\pi^-) = -\rho(\pi^+) = (-1, 0, 0, 0, 0)$ . Thus, for  $V_\alpha$  where ( $\alpha = \pi^-, \pi^+, K^+, K^0, K^-, \bar{K}^0, \bar{D}^0, \dots, B^+, \dots, \bar{T}^0, \dots, T_b^+$ ), utilizing Eq. (2), Table 1 (defines the  $V_\alpha$  in terms of the  $\lambda$  matrices), and Table 3 (which lists all nonzero commutators), one can extract the positive root vectors  $\rho(\alpha)$ . One can also obtain the positive roots by using:

$$\text{positive roots are given by } \mathbf{m}(i) - \mathbf{m}(j) \quad \text{for } i < j. \quad (8)$$

Either way, we obtain the following positive root listing — note that the positive roots lie to the right of the diagonal of the  $6 \times 6$  Lie algebra generator matrix in Eq. (6) and have length 1:

List of positive root vectors		
$\rho(\pi^+) = (1, 0, 0, 0, 0)$	$\rho(K^+) = (\frac{1}{2}, \frac{3}{2\sqrt{3}}, 0, 0, 0)$	$\rho(\bar{D}^0) = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}}, 0, 0)$
$\rho(B^+) = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}, 0)$	$\rho(\bar{T}^0) = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{3}{\sqrt{15}})$	$\rho(K^0) = (-\frac{1}{2}, \frac{3}{2\sqrt{3}}, 0, 0, 0)$
$\rho(D^-) = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}}, 0, 0)$	$\rho(B^0) = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}, 0)$	$\rho(T^-) = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{3}{\sqrt{15}})$
$\rho(D_s^-) = (0, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, 0, 0)$	$\rho(B_s^0) = (0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}, 0)$	$\rho(T_s^-) = (0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{3}{\sqrt{15}})$
$\rho(B_c^+) = (0, 0, -\frac{3}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}, 0)$	$\rho(\bar{T}_c^0) = (0, 0, -\frac{3}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{3}{\sqrt{15}})$	$\rho(T_b^-) = (0, 0, 0, -\frac{2}{\sqrt{10}}, \frac{3}{\sqrt{15}})$

Of the 15 positive roots, only  $l = 5$  are linearly independent and complete and are called the simple roots. The number of simple roots is equal to the rank of the algebra, the number of Cartan generators. The simple roots are

$$\rho(\alpha_i) = \mathbf{m}(i) - \mathbf{m}(i + 1) \quad \text{for } i = 1, \dots, (6 - 1) = 5. \quad (9)$$

List of simple root vectors		
$\rho(\pi^+) = (1, 0, 0, 0, 0)$	$\rho(K^0) = (-\frac{1}{2}, \frac{3}{2\sqrt{3}}, 0, 0, 0)$	$\rho(D_s^-) = (0, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, 0, 0)$
$\rho(B_c^+) = (0, 0, -\frac{3}{2\sqrt{6}}, \frac{5}{2\sqrt{10}}, 0)$	$\rho(T_b^-) = (0, 0, 0, -\frac{2}{\sqrt{10}}, \frac{3}{\sqrt{15}})$	

### 1.3. Fundamental weights

The fundamental representation weights  $\mu_j$  of  $SU(6)$  are given by

$$\frac{2\rho(\alpha_i) \cdot \mu_j}{\alpha_i^2} = \delta_{ij}, \quad (10)$$

$$\mu_j = \sum_{k=1}^j \mathbf{m}(k). \quad (11)$$



List of  $SU(6)$  fundamental weights

$$\begin{aligned} \boldsymbol{\mu}_1 &= \left( \frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{15}} \right) & \boldsymbol{\mu}_2 &= \left( 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}} \right) & \boldsymbol{\mu}_3 &= \left( 0, 0, \frac{3}{2\sqrt{6}}, \frac{3}{2\sqrt{10}}, \frac{3}{2\sqrt{15}} \right) \\ \boldsymbol{\mu}_4 &= \left( 0, 0, 0, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{15}} \right) & \boldsymbol{\mu}_5 &= \left( 0, 0, 0, 0, \frac{5}{2\sqrt{15}} \right) \end{aligned}$$

The rank  $l$   $\boldsymbol{\mu}_j$  are the highest (dominant) weights of the rank  $l$  fundamental representations and are complete. Thus, the highest weight  $\boldsymbol{\mu}$  of *any* irreducible  $SU(6)$  representation can be written in terms of these  $\boldsymbol{\mu}_j$ . Indeed, we have

$$\frac{2\rho(\alpha_j) \cdot \boldsymbol{\mu}}{\alpha_j^2} = l_j, \quad (12)$$

$$\boldsymbol{\mu} = \sum_{j=1}^l l_j \boldsymbol{\mu}_j, \quad (13)$$

where the  $l_j$  are Dynkin coefficients and are non-negative integers. The  $SU(6)$  35-plet is denoted by  $(1, 0, 0, 0, 1)$ . The *standard* Young tableau can be constructed by noting that the  $k$ th Dynkin label is the number of tableau columns with  $k$  boxes. Thus, for instance,  $(1, 1, 0, 0, 0) \sim \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ . In general (see Ref. 9), a tableau is specified by  $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$  (here the  $p_j =$  positive integers  $= l_j$  ( $j = 1, \dots, l$ ) in Eq. (13)) and the  $i$ th *fundamental representation* is given by

$$p_i = 1, \quad p_j = 0, \quad \text{where } i \neq j \quad \text{for } i, j = 1, \dots, (6-1) = 5. \quad (14)$$

Thus,  $(1, 0, 0, 0, 0) \sim \square \sim$  first fundamental representation of  $SU(6)$ . We also note that for an irreducible representation (irrep.) with  $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$ , the highest weight is given by

$$\boldsymbol{\mu}_{\text{dominant weight}} = \sum_{i=1}^l p_i \boldsymbol{\mu}_i. \quad (15)$$

So the highest weight for the irrep.  $(1, 1, 0, 0, 0)$  is  $\left( \frac{1}{2}, \frac{3}{2\sqrt{3}}, \frac{3}{2\sqrt{6}}, \frac{3}{2\sqrt{10}}, \frac{3}{2\sqrt{15}} \right)$ , whereas the highest weight for the irrep.  $(1, 0, 0, 0, 1)$  is  $\left( \frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \frac{1}{2\sqrt{10}}, \frac{3}{\sqrt{15}} \right)$ .

The dimension  $D_6$  of a particular representation with  $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$  is given by the following equation (see Ref. 9):

$$\begin{aligned} & D_6(p_1, p_2, p_3, p_4, p_5) \\ &= \frac{1}{2!3!4!5!} (p_1 + 1)(p_1 + p_2 + 2)(p_1 + p_2 + p_3 + 3) \\ & \quad \times (p_2 + 1)(p_2 + p_3 + 2)(p_3 + 1)(p_4 + 1)(p_4 + p_3 + 2) \\ & \quad \times (p_4 + p_3 + p_2 + 3)(p_4 + p_3 + p_2 + p_1 + 4)(p_5 + 1)(p_5 + p_4 + 2) \\ & \quad \times (p_5 + p_4 + p_3 + 3)(p_5 + p_4 + p_3 + p_2 + 4)(p_5 + p_4 + p_3 + p_2 + p_1 + 5). \quad (16) \end{aligned}$$

## 2. Equal-Time Commutation and Anticommutation Relations and Infinite-Momentum Frame Asymptotic Symmetry

$SU(6)$  35-plet  $q\bar{q}$  representation matrix

$$\begin{array}{c}
 \bar{u} \quad \bar{d} \quad \bar{s} \quad \bar{c} \quad \bar{b} \quad \bar{t} \\
 \begin{array}{l}
 u \\
 d \\
 s \\
 c \\
 b \\
 t
 \end{array}
 \begin{pmatrix}
 u\bar{u} & u\bar{d} & u\bar{s} & u\bar{c} & u\bar{b} & u\bar{t} \\
 d\bar{u} & d\bar{d} & d\bar{s} & d\bar{c} & d\bar{b} & d\bar{t} \\
 s\bar{u} & s\bar{d} & s\bar{s} & s\bar{c} & s\bar{b} & s\bar{t} \\
 c\bar{u} & c\bar{d} & c\bar{s} & c\bar{c} & c\bar{b} & c\bar{t} \\
 b\bar{u} & b\bar{d} & b\bar{s} & b\bar{c} & b\bar{b} & b\bar{t} \\
 t\bar{u} & t\bar{d} & t\bar{s} & t\bar{c} & t\bar{b} & t\bar{t}
 \end{pmatrix}.
 \end{array} \tag{17}$$

$SU(6)$  normalized, orthogonal, and traditional zero weight particle representation states constructed using diagonal  $SU(6)$  group quark matrix elements

$$\begin{aligned}
 |\eta_3\rangle &= \left| \frac{u\bar{u} - d\bar{d}}{\sqrt{2}} \right\rangle = |\pi^0\rangle, \\
 |\eta_8\rangle &= \left| \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}} \right\rangle \sim |\eta\rangle, \\
 |\eta_{15}\rangle &= \left| \frac{u\bar{u} + d\bar{d} + s\bar{s} - 3c\bar{c}}{2\sqrt{3}} \right\rangle, \\
 |\eta_{24}\rangle &= \left| \frac{u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c} - 4b\bar{b}}{2\sqrt{5}} \right\rangle, \\
 |\eta_{35}\rangle &= \left| \frac{u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c} + b\bar{b} - 5t\bar{t}}{\sqrt{30}} \right\rangle, \\
 |\eta_0\rangle &= \left| \frac{u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c} + b\bar{b} + t\bar{t}}{\sqrt{6}} \right\rangle.
 \end{aligned} \tag{18}$$

In Ref. 22, using infinite-momentum frame broken asymptotic symmetry, we calculated the magnetic moments of the *physical* on-mass shell  $J^P = 3/2^+$  ground-state decuplet baryons without ascribing any specific form to their quark structure or intra-quark interactions by using *equal-time commutation relations* (ETCRs) which involve at most one current density, thus, avoiding problems associated with Schwinger terms. Here, the ETCRs involve the vector charge generators (the  $V_\alpha$ ) of the symmetry groups of QCD. They are valid even though these symmetries are broken<sup>3,14,15,22–27,33–35</sup> and even when the Lagrangian is not known or cannot be constructed.

As shown in Ref. 22 and references therein, infinite-momentum frame broken asymptotic symmetry is characterized by the existence of physical on-mass-shell hadron annihilation operators  $a_\alpha(\mathbf{k}, \lambda)$  (momentum  $\mathbf{k}$  ( $|\mathbf{k}| \rightarrow \infty$ ), helicity  $\lambda$ , and  $SU_F(N)$  flavor index  $\alpha$ ) and their creation operator counterparts which produce physical states when acting on the vacuum. Indeed, the *physical* on-mass-shell

hadron annihilation operator  $a_\alpha(\mathbf{k}, \lambda)$  is related linearly under flavor transformations to the *representation* annihilation operator  $a_j(\mathbf{k}, \lambda)$ . Thus, in the infinite-momentum frame, physical states denoted by  $|\alpha, \mathbf{k}, \lambda\rangle$  (which do not belong to irreducible representations) are linear combinations of representation states denoted by  $|j, \mathbf{k}, \lambda\rangle$  (which do belong to irrep.) plus nonlinear corrective terms that are best calculated in a frame where mass differences are deemphasized such as in the infinite-momentum frame. Mathematically,<sup>14,15,22,23,27–30</sup> this is expressed by  $|\alpha, \mathbf{k}, \lambda\rangle = \sum_j C_{\alpha j} |j, \mathbf{k}, \lambda\rangle$ ,  $|\mathbf{k}| \rightarrow \infty$ , where the orthogonal matrix  $C_{\alpha j}$  depends on *physical*  $SU_F(N)$  mixing parameters, is defined *only* in the  $\infty$ -momentum frame, and can be constrained directly by ETCRs.

It cannot be overemphasized that the particular Lorentz frame that one utilizes when analyzing current-algebraic sum rules does not matter when flavor symmetry is exact and is strictly a matter of taste and calculational convenience, whereas when one uses current-algebraic sum rules in broken symmetry, the choice of frame is *paramount* since one wishes to emphasize the calculation of leading order contributions while simultaneously simplifying the calculation of symmetry breaking corrections.<sup>14,15,22–27,31–33</sup>

While we will only discuss the  $J^{PC} = 0^{-+}$  35-plet representation in this paper, nevertheless, it is instructive to outline our normalization conventions including fermionic representation states. In Table 5, we give all nonzero anticommutation relations, where the singlet  $U(1)$  matrix  $V_0$  is explicitly present. We have particle four-momentum  $p = (p^0, \mathbf{p})$ , with

$$\begin{aligned} [a^{(r)}(p), a^{\dagger(s)}(p')]_+ &= [b^{(r)}(p), b^{\dagger(s)}(p')]_+ = N_a \delta_{rs} \delta^3(\mathbf{p} - \mathbf{p}'), \\ u^{\dagger(r)}(p) u^{(s)}(p) &= N_D \delta_{rs}, \end{aligned}$$

$$\psi(x) = \sum_r \int d^3p N_\psi [a^{(r)}(p) u^{(r)}(p) e^{-ip \cdot x} + b^{\dagger(r)}(p) v^{(r)}(p) e^{+ip \cdot x}],$$

$$a^{(r)}(p) = ((2\pi)^3 N_\psi N_D)^{-1} \int d^3x e^{+ip \cdot x} \bar{u}^r(p) \gamma_0 \psi(x),$$

$$b^{(r)}(p) = ((2\pi)^3 N_\psi N_D)^{-1} \int d^3x e^{+ip \cdot x} \bar{\psi}(x) \gamma_0 v^{(r)}(p),$$

$$\langle s', \mathbf{p}', \lambda' | s, \mathbf{p}, \lambda \rangle = \delta_{ss'} \delta_{\lambda\lambda'} (2\pi)^3 2p^0 \delta^3(\mathbf{p}' - \mathbf{p}),$$

where  $a^{(r)}(p)$  and  $b^{(r)}(p)$  are creation operators,  $u^{(s)}(p)$ ,  $v^{(r)}(p)$  are Dirac spinors,  $\psi(x)$  is a spin 1/2 Dirac field operator ( $s, \lambda$  denote particle spin and helicity, respectively),  $|\psi\rangle = \sum_{s,\lambda} \int N_a^{-1} d^3p |s, p, \lambda\rangle \langle s, p, \lambda | \psi \rangle$ ,  $(2\pi)^6 N_\psi^2 N_D^2 \langle N | \Omega^{\mu \cdots} | N \rangle$  is covariant (transforms like  $\Omega^{\mu \cdots}$ ), and  $(2\pi)^3 N_\psi^2 N_D N_a = 1$ .

### 3. The Physical Electromagnetic Current

We now discuss the vector charges and two-particle basis states after imposing a unitary homogeneous pure Lorentz transformation  $-\hat{z}$  boost such that the

three-momentum  $\mathbf{k}$  of all states has  $|\mathbf{k}| \rightarrow \infty$ , and all creation operators produce *physical* states when acting on the physical vacuum (see Refs. 22 and 23 for more details).

First, we note that the charges operating on states (see Eqs. (6), (17) and (18)) transform according to

$$V_i^j |q_k \bar{q}_l\rangle = \delta_k^j |q_i \bar{q}_l\rangle - \delta_i^l |q_k \bar{q}_j\rangle. \quad (19)$$

Bras and kets are given by

$$\begin{aligned} |D^+, p'\rangle (J^{PC} = 0^{-+}) &= |c\bar{d}\rangle = |u_4 \bar{u}_2\rangle, \\ \langle D^{*+}, p'| (J^{PC} = 1^{--}) &\equiv \langle {}^*c\bar{d}| = \langle {}^*u_4 \bar{u}_2|, \text{ etc.}, \end{aligned} \quad (20)$$

where the spacelike four-momentum transfer  $q^2$  is given by

$$q^2 = (p' - p)^2 = m^{*2} + m^2 - 2p' \cdot p, \quad (21)$$

$$p' = (E', \mathbf{s}) = \left( \sqrt{m^{*2} + s_x^2 + s_z^2}, s_x, 0, s_z \right), \quad (22)$$

$$p = (E, \mathbf{t}) = \left( \sqrt{m^2 + t_z^2}, 0, 0, t_z \right), \quad (23)$$

$$\text{Set } s_z = rt_z \quad \text{and} \quad 0 < r = \text{const}, \quad (24)$$

$$p' \cdot p = \sqrt{m^{*2} + s_x^2 + r^2 t_z^2} \sqrt{m^2 + t_z^2} - rt_z^2, \quad (25)$$

$$p' \cdot p = rt_z^2 \sqrt{1 + \frac{m^{*2} + s_x^2}{r^2 t_z^2}} \sqrt{1 + \frac{m^2 + t_z^2}{t_z^2}} - rt_z^2, \quad (26)$$

$$\overbrace{p' \cdot p}^{t_z \rightarrow \infty} = \frac{1}{2} \left( \frac{m^{*2} + s_x^2 + r^2 m^2}{r} \right), \quad (27)$$

$$q^2 = m^{*2} + m^2 - \left( \frac{m^{*2} + s_x^2 + r^2 m^2}{r} \right), \quad (28)$$

$$q^2 = \frac{-(1-r)}{r} m^{*2} \left[ 1 - \left( \frac{m^2}{m^{*2}} \right) r \right] - \frac{s_x^2}{r}. \quad (29)$$

The physical vector charge  $V_{K^0}$  is  $V_{K^0} = V_6 + iV_7$ , the physical vector charge  $V_{\pi^\pm} = V_1 \pm iV_2$ , etc. The  $\lambda_a$ ,  $a = 1, 2, \dots, 35$  satisfy the Lie algebra  $[(\lambda_a/2), (\lambda_b/2)] = i \sum_c f_{abc} (\lambda_c/2)$ , where the  $f_{abc}$  are structure constants of the flavor group  $SU_F(6)$  and  $V_a^\mu(x) = \bar{q}^i(x) (\lambda_a/2)_{ij} \gamma^\mu q^j(x)$ .

The *physical electromagnetic current*  $j_{\text{em}}^\mu(0)$  ( $u, d, s, c, b, t$  quark system) is

$$\begin{aligned} j_{\text{em}}^\mu(0) &= V_3^\mu(0) + \left( \frac{1}{3} \right)^{1/2} V_8^\mu(0) - \left( \frac{2}{3} \right)^{1/2} V_{15}^\mu(0) \\ &+ \left( \frac{2}{5} \right)^{1/2} V_{24}^\mu(0) - \left( \frac{3}{5} \right)^{1/2} V_{35}^\mu(0) + \left( \frac{1}{3} \right)^{1/2} \times V_0^\mu(0) \end{aligned} \quad (30)$$

$$= j_V^\mu(0) + j_S^\mu(0), \quad (31)$$

where  $j_V^\mu(0) \equiv j_{\text{em}3}^\mu(0)$  = the iso-vector part of the electromagnetic current,  $j_S^\mu(0) \equiv$  the isoscalar part of the electromagnetic current. The flavor  $U(1)$  singlet current  $V_0^\mu(x) = \bar{q}^i(x)(\lambda_0/2)_{ij}\gamma^\mu q^j(x)$  where  $\lambda_0 \equiv \sqrt{1/3}I$ ,  $I$  is the identity, so that  $\text{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$  holds for all  $\lambda_{a'}$  ( $a' = 0, 1, 2, \dots, 35$ ), and  $j_0^\mu = V_0^\mu/\sqrt{3}$ . The  $U(1)$  singlet charge  $V_0$  commutes with all of the  $V_a$ .

From the commutation relations in Table 3, we obtain the fascinating equation (it contains only ETCRs and explicitly the electromagnetic current singlet  $j_0^\mu$ ) — in broken symmetry  $j^\mu \equiv j_{\text{em}}^\mu(0)$  (momentum  $\mathbf{k}$  with  $|\mathbf{k}| \rightarrow \infty$ ):

$$\begin{aligned} & [[j^\mu, V_{\pi^+}], V_{\pi^-}] + [[j^\mu, V_{D_s^-}], V_{D_s^+}] + [[j^\mu, V_{T_b^-}], V_{T_b^+}] \\ &= 2j^\mu - 2\left(\frac{V_0^\mu}{\sqrt{3}}\right) = 2j^\mu - 2j_0^\mu. \end{aligned} \quad (32)$$

The angles between the simple roots appearing in Eq. (32) are:  $\theta_{\rho(\pi^+), \rho(D_s^-)} = \theta_{\rho(D_s^-), \rho(T_b^-)} = 90^\circ$ . These angles correspond to the  $(u, d)$ ,  $(c, s)$ , and  $(t, b)$  doublet “sectors.”

For vector ( $V$ )  $\rightarrow$  pseudoscalar ( $P$ ) radiative decays, we have (in the infinite-momentum frame [ $t_z \rightarrow \infty$ , see Eq. (29)],  $\lambda = 1 =$  vector meson polarization index,  $\mu = 0$ ,  $r = 1$ , and  $s_x^2 = 0 \Rightarrow q^2 = 0$  in the following matrix element):

$$\langle V|j^\mu|P\rangle = \langle V\rangle\epsilon^{\mu\nu\rho\sigma}\epsilon_\nu(\mathbf{p}, \lambda)p'_\rho p_\sigma, \quad (33a)$$

$$\Gamma(V(p', \lambda = 1) \rightarrow P(p) + \gamma(q^2 = 0)) = \frac{\langle V\rangle^2}{96\pi} \left(\frac{m_V^2 - m_P^2}{m_V}\right)^3, \quad (33b)$$

$$|\langle V\rangle| = \left[96\pi \left(\frac{m_V}{m_V^2 - m_P^2}\right)^3 \Gamma(V \rightarrow P + \gamma)\right]^{\frac{1}{2}}. \quad (33c)$$

So for instance, if we evaluate Eq. (32) between the states  $\langle D^{*+}|$  and  $|D^+\rangle$  using Eqs. (17), (19) and (20) we obtain

$$\langle D^{*0}\rangle + \langle \bar{K}^{*0}\rangle = 2\langle D^{*+}\rangle_0, \quad (34)$$

where

$$\langle D^{*0}\rangle \equiv \langle D^{*0}|j^\mu|D^0\rangle,$$

$$\langle \bar{K}^{*0}\rangle \equiv \langle \bar{K}^{*0}|j^\mu|K^0\rangle,$$

$$\langle \rho^+\rangle_0 \equiv \langle \rho^+|\frac{1}{\sqrt{3}}V_0^\mu|\pi^+\rangle,$$

$$\langle D^{*0}\rangle_0 \equiv \langle D^{*0}|\frac{1}{\sqrt{3}}V_0^\mu|D^0\rangle, \text{ etc.}$$

We define

$$X_{1--} \equiv \begin{pmatrix} \rho^0 \\ \omega \\ \phi \\ \psi \\ \Upsilon \\ t\bar{t} \end{pmatrix}, \quad (35)$$

$$X_{0-+} \equiv \begin{pmatrix} \pi^0 \\ \eta \\ \eta' \\ \eta_c \\ \eta_b \\ \eta_t \end{pmatrix}. \quad (36)$$

Evaluating Eq. (32) between the 16 bra-ket state pairs with bras yields

$$\begin{aligned} &\langle \rho^0 |, \langle \rho^+ |, \langle K^{*+} |, \langle \bar{D}^{*0} |, \langle B^{*+} |, \langle \bar{T}^{*0} |, \langle K^{*0} |, \langle D^{*-} |, \\ &\langle B^{*0} |, \langle T^{*-} |, \langle D_s^{*-} |, \langle B_s^{*0} |, \langle T_s^{*-} |, \langle B_c^{*+} |, \langle \bar{T}_c^{*0} | \quad \text{and} \quad \langle \bar{T}_b^{*-} |, \end{aligned}$$

then we find that

$$\langle \rho^+ \rangle = \langle \rho^- \rangle = \langle \rho^0 \rangle_0, \quad (37)$$

$$\langle \rho^0 \rangle = \langle \rho^+ \rangle_0, \quad (38)$$

$$\langle K^{*0} \rangle + \langle \bar{D}^{*0} \rangle = 2\langle K^{*+} \rangle_0, \quad (39)$$

$$\langle K^{*+} \rangle + \langle D^{*-} \rangle = 2\langle \bar{D}^{*0} \rangle_0, \quad (40)$$

$$\langle B^{*0} \rangle + \langle \bar{T}^{*0} \rangle = 2\langle B^{*+} \rangle_0, \quad (41)$$

$$\langle B^{*+} \rangle + \langle T^{*-} \rangle = 2\langle \bar{T}^{*0} \rangle_0, \quad (42)$$

$$\langle K^{*+} \rangle + \langle D^{*-} \rangle = 2\langle K^{*0} \rangle_0, \quad (43)$$

$$\langle K^{*0} \rangle + \langle \bar{D}^{*0} \rangle = 2\langle D^{*-} \rangle_0, \quad (44)$$

$$\langle B^{*+} \rangle + \langle T^{*-} \rangle = 2\langle B^{*0} \rangle_0, \quad (45)$$

$$\langle B^{*0} \rangle + \langle \bar{T}^{*0} \rangle = 2\langle T^{*-} \rangle_0, \quad (46)$$

$$\begin{aligned} &\langle D_s^{*-} \rangle \langle D_s^- | V_{D_s^-} | [X_{0-+} \rangle \langle X_{0-+} | | V_{D_s^+}^+ | D_s^- \rangle \\ &\quad - \langle D_s^{*-} | V_{D_s^-} | [X_{1--} \rangle \langle X_{1--} | | j^\mu | [X_{0-+} \rangle \langle X_{0-+} | | V_{D_s^+}^+ | D_s^- \rangle \\ &= 2\langle D_s^{*-} \rangle - 2\langle D_s^{*-} \rangle_0, \end{aligned} \quad (47)$$

$$\langle B_c^{*+} \rangle + \langle T_s^{*-} \rangle = 2\langle B_s^{*0} \rangle_0, \quad (48)$$

$$\langle B_s^{*0} \rangle + \langle \bar{T}_c^{*0} \rangle = 2\langle T_s^{*-} \rangle_0, \quad (49)$$

$$\langle B_s^{*0} \rangle + \langle \bar{T}_c^{*0} \rangle = 2\langle B_c^{*+} \rangle_0, \quad (50)$$

$$\langle B_c^{*+} \rangle + \langle \bar{T}_s^{*-} \rangle = 2\langle \bar{T}_c^{*0} \rangle_0, \quad (51)$$

$$\begin{aligned} & \langle T_b^{*-} \rangle \langle T_b^- | V_{T_b^-}^- | [X_{0-+}] \langle X_{0-+} \rangle | V_{T_b^+}^+ | T_b^- \rangle \\ & - \langle T_b^{*-} | V_{T_b^-}^- | [X_{1--}] \langle X_{1--} \rangle | j^\mu | [X_{0-+}] \langle X_{0-+} \rangle | V_{T_b^+}^+ | T_b^- \rangle \\ & = 2\langle T_b^{*-} \rangle - 2\langle T_b^{*-} \rangle_0. \end{aligned} \quad (52)$$

Equations (37)–(52) *explicitly demonstrate in broken symmetry* the importance of the electromagnetic current singlet  $U(1)$  matrix  $V_0$  contribution to radiative decays. Indeed, one finds that  $\Gamma(\rho^\pm \rightarrow \pi^\pm \gamma)$  and  $\Gamma(\rho^0 \rightarrow \pi^0 \gamma)$  are *entirely due to the electromagnetic current singlet contribution*. At present, insufficient data are available for most of the decay matrix elements in Eqs. (37)–(52). Even where there are data, the signs of the matrix elements are not yet experimentally available, although there exist theoretical models which predict matrix element signs.<sup>1</sup> From Eqs. (39) and (44), we find that  $\langle D^{*-} \rangle_0 = \langle K^{*+} \rangle_0$ . Similarly, we find from Eqs. (40) and (43), we find that  $\langle \bar{D}^{*0} \rangle_0 = \langle K^{*0} \rangle_0$ . Very little is known about the behavior of the singlet generator in broken symmetry, other than that given by Eqs. (37)–(52) — to partially remedy that situation let the right-hand sides of Eqs. (39) and (43) be proportional, i.e.  $\langle K^{*0} \rangle_0 = \beta \langle K^{*+} \rangle_0$ . We then obtain<sup>5</sup>

$$\frac{\langle D^{*0} \rangle}{\langle K^{*+} \rangle} = \frac{1}{\beta} * \left[ 1 + \frac{\langle D^{*-} \rangle}{\langle K^{*+} \rangle} - \beta * \frac{\langle K^{*0} \rangle}{\langle K^{*+} \rangle} \right], \quad \text{where } \beta \neq 0. \quad (53)$$

From data in Ref. 5, we get ( $\pm$  signs are not correlated and  $\beta$  is assumed to be 1) (an assumption suggested only by Eqs. (37) and (38)  $\rho$  triplet charged and neutral singlet results and perhaps holding for doublets as well):

$$\begin{aligned} \frac{\langle D^{*0} \rangle}{\langle K^{*+} \rangle} &= 1 + (\pm(0.56 \pm 0.08)) - (\pm(1.52 \pm 0.10)) \\ &= \begin{cases} +0.05 \pm 0.13 & \text{for } ++, \\ +3.06 \pm 0.13 & \text{for } +-, \\ -1.06 \pm 0.13 & \text{for } -+, \\ +1.95 \pm 0.13 & \text{for } --. \end{cases} \end{aligned} \quad (54)$$

This implies that (statistical propagation of errors — quadrature calculated):

$$\Gamma(D^{*0} \rightarrow D^0 \gamma)_{\beta=1} = \begin{cases} 0.13_{-0.13}^{+0.65} \text{ keV} & \text{for } ++, \\ 468.0 \pm 61.0 \text{ keV} & \text{for } +-, \\ 56.0 \pm 15.0 \text{ keV} & \text{for } -+, \\ 189.0 \pm 31.0 \text{ keV} & \text{for } --. \end{cases} \quad (55)$$

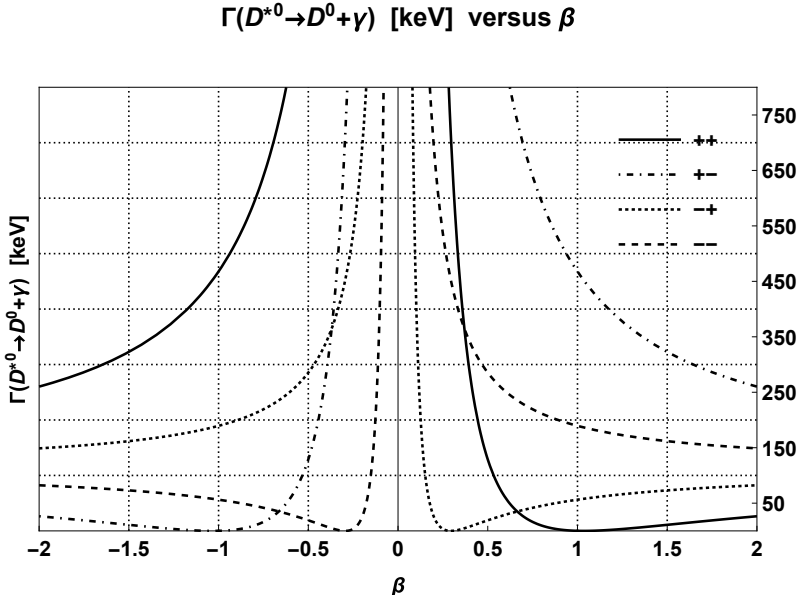


Fig. 1.  $\Gamma(D^{*0} \rightarrow D^0 + \gamma)$  versus  $\beta$ .

In Fig. 1, we graphically show  $\Gamma(D^{*0} \rightarrow D^0 + \gamma)$  versus  $\beta$  as  $\beta$  is allowed to vary from  $-2$  to  $2$ .

From Fig. 1, we see that  $\Gamma(D^{*0} \rightarrow D^0 \gamma)$  is dependent upon the signs of  $\frac{\langle D^{*-} \rangle}{\langle K^{*+} \rangle}$  and  $\frac{\langle K^{*0} \rangle}{\langle K^{*+} \rangle}$ . However, in unbroken  $SU(3)$   $\frac{\langle K^{*0} \rangle}{\langle K^{*+} \rangle}$  is negative and  $= -2$ .<sup>1,15,26</sup> From experiment we therefore choose  $\frac{\langle K^{*0} \rangle}{\langle K^{*+} \rangle} = -1.51 \pm 0.10$ , so we expect the  $+ -$  or  $- -$  curves in Fig. 1 to be more physically predictive. From Table 3, we have at our disposal

$$[V_{K^0}, j^\mu] = 0, \quad (56a)$$

$$[V_{D^0}, j^\mu] = 0. \quad (56b)$$

Evaluating (we neglect intermultiplet mixing with the  $K^*(1410)$ ) Eq. (56a) between the physical asymptotic states  $\langle K^+ |$  and  $|\rho^+ \rangle$  and Eq. (56b) between the physical asymptotic states  $\langle D^+ |$  and  $|\rho^+ \rangle$ , one obtains

$$\langle D^{*+} \rangle = \langle K^{*+} \rangle = \langle \rho^+ \rangle. \quad (57)$$

Thus, we expect (conjugate states used) that  $\frac{\langle D^{*-} \rangle}{\langle K^{*+} \rangle}$  is positive and the  $+ -$  curve in Fig. 1, to be that which is operative. An experimental determination of  $\Gamma(D^{*0} \rightarrow D^0 \gamma)$  would then provide a value for  $\beta$  which may be useful in further research — especially where higher rank special unitary groups and Lie algebras play a role. Unfortunately, current experimental data from Ref. 5 yields only that  $\Gamma(D^{*0} \rightarrow D^0 \gamma) \leq 741.3$  keV, CL = 90%.



At present, quantum field theories (including SUSY theories) have not been successful in replacing the standard model (QCD... which does not include gravity) with a grand unified theory without hierarchical or other problems. Generally speaking, most theories are perturbative and renormalizable with local gauge fields strongly related to Lie algebras and utilize spontaneous symmetry breaking. In Lie algebraic representations of interest, anomalies (for instance, see Refs. 11, 36, 37, especially Chap. 22) and, Refs. 38 and 39 must vanish for physical representations — a severe constrain on those theories. On the other hand, Eqs. (37)–(52) are *nonperturbative*.

#### 4. Summary and Conclusions

We presented research on radiative decays of vector ( $J^{PC} = 1^{--}$ ) to pseudoscalar ( $J^{PC} = 0^{-+}$ ) particles ( $u, d, s, c, b, t$  quark system) using broken symmetry techniques in the infinite-momentum frame and equal-time commutation relations. The research utilized the  $SU(6)$  Lie algebra characterization of flavor  $SU_F(6)$  representations and the *physical electromagnetic current*  $j_{\text{em}}^\mu(0)$  including its singlet  $U(1)$  term and focused on the 35-plet. The research was conducted without ascribing any specific form to meson quark structure or intra-quark interactions by using ETCRs which involve at most one current density, thus, *avoiding problems associated with Schwinger terms*. We found that *the electromagnetic current singlet plays an intrinsic role in understanding the physics of radiative decays* where (bilinear) commutators of the  $SU(6)$  Lie algebra generators are Lie products acting over the real number field. Indeed, in broken symmetry and the infinite-momentum frame, we developed a new and fascinating equation involving the electromagnetic current (including its singlet-proportional to the  $SU(6)$  singlet), three  $SU(6)$  simple roots, and double commutators using ETCRs.

For notational conciseness and self-containment and use by other researchers,  $SU(6)$  Lie algebra simple roots, positive roots, weights, fundamental weights, nonzero commutators, and nonzero anticommutators were also determined which allow construction of all  $SU(6)$  representations. Surprisingly — after symmetry breaking — we discovered that charged and neutral  $\rho$  meson radiative decays into  $\pi\gamma$  were due entirely to the singlet term in  $j_{\text{em}}^\mu(0)$ . Although there is insufficient experimental data on the radiative decay  $\Gamma(D^{*0} \rightarrow D^0\gamma)$  available at this time, we derived equations involving *physical* matrix elements of the  $SU(6)$  singlet generator which allowed parametrization of possible predicted values of  $\Gamma(D^{*0} \rightarrow D^0\gamma)$  versus  $\beta$ .

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